Differential Geometry I: Curves and surfaces

G. Moschidis

14 Nov. 2025

10.1 Let $\mathcal{M} \subset \mathbb{R}^n$ be an m dimensional submanifold of class C^k and let $\mathcal{U} \subset \mathbb{R}^n$ be an open set such that $\mathcal{M} \cap \mathcal{U} \neq \emptyset$. Show that $\mathcal{M} \cap \mathcal{U}$ is also an m dimensional submanifold of class C^k.

Solution. This is a simple application of the definition of a submanifold. Let $p \in \mathcal{M} \cap \mathcal{U}$. Recall that, since $\mathcal{M}$ is a C^k submanifold of $\mathbb{R}^n$, there exists an open set $\mathcal{V} \subset \mathbb{R}^n$ with $p \in \mathcal{V}$ and a C^k diffeomorphism $\phi : \mathcal{V} \to \mathcal{V}' \subset \mathbb{R}^n$ and an m dimensional vector subspace $E \subset \mathbb{R}^n$ such that $\phi(\mathcal{M} \cap \mathcal{V}) = E \cap \mathcal{V}'$. Let us now set $\mathcal{W} = \mathcal{V} \cap \mathcal{U}$ (this is a non-empty neighborhood of $p \in \mathcal{M} \cap \mathcal{U}$, since both open sets contain p) and $\mathcal{W}' = \phi(\mathcal{V} \cap \mathcal{U})$. Then $\psi = \phi_{\mathcal{V}\cap\mathcal{U}} : \mathcal{W} \to \mathcal{W}'$ is a diffeomorphism and satisfies for any $B \subset \mathcal{V}$:

$$\psi(B \cap \mathcal{U}) = \phi(B) \cap \mathcal{W}'$$

(that's an easy exercise). In particular, $\psi(\mathcal{M} \cap \mathcal{W}) = \phi_{\mathcal{V}\cap\mathcal{U}}(\mathcal{M} \cap (\mathcal{V} \cap \mathcal{U})) = E \cap \mathcal{W}'$. Hence, $\mathcal{M} \cap \mathcal{U}$ is a C^k submanifold of $\mathbb{R}^n$ of dimension m.

10.2 **(a)** First, let's recall some notions from linear algebra: Explain why the volume of the parallelepiped $P \subset \mathbb{R}^m$ constructed on the vectors $b_1, \dots, b_m \in \mathbb{R}^m$ satisfies

$$\mathrm{Vol}(P) = \sqrt{\det(G)},$$

where G is the Gram matrix of $b_1, \dots, b_m$ (i.e., the matrix whose coefficients are the inner products $g_{ij} = \langle b_i, b_j \rangle$).

(b) Let $\psi : \Omega \to \mathcal{M}$ be a C^1 local parametrization of the submanifold $\mathcal{M} \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^m$ is open and let G be the associated metric tensor. Informally explain (using the previous part) why $\sqrt{\det(G(u))}du_1 \dots du_m$ is "reasonable" as an expression for the volume of the infinitesimal coordinate parallelepiped around $\psi(u)$ on $\mathcal{M}$.

(c) Show that the volume of $\psi(\Omega)$, namely

$$\int_\Omega \sqrt{\det(G(u))}du_1 \dots du_m$$

is invariant under changes of parametrization, i.e. if $h : \Omega' \to \Omega$ is a C^1 diffeomorphism and G' is the metric tensor associated to $\psi' = \psi \circ h$, then

$$\int_{\Omega'} \sqrt{\det(G'(u'))}du'_1 \dots du'_m = \int_\Omega \sqrt{\det(G(u))}du_1 \dots du_m.$$

Solution. (a) In the planar case $m = 2$, the area (i.e. 2 dimensional volume) of a parallelogram is

$$\mathrm{Area}(P) = \sqrt{\|b_1\|^2\|b_2\|^2 - \langle b_1, b_2 \rangle^2}.$$

In general, the oriented volume of parallelepiped P is the determinant of the matrix B whose columns are the coordinate vectors of the b_i in the canonical basis:

$$\mathrm{Vol}_{\mathrm{or}}(P) = \det(B), \qquad B = (b_{ij}), \quad b_j = \sum_{i=1}^n b_{ij} e_i.$$

But since $G = B^\top B$, we have

$$\det(G) = \det(B^\top B) = \det(B^\top)\det(B) = \det(B)^2 = \mathrm{Vol}(P)^2.$$

(b) If we formally denote by $du = (du_1, \dots, du_n)$ an infinitesimal line segment in the Ω plane (connecting the infinitessimally close points u and $u+du$), then the image of that through ψ (namely the infinitesimal segment connecting $\psi(u)$ to $\psi(u+du)$) is given by $d\psi(u) = \sum_{i=1}^n \frac{\partial\psi}{\partial u_i} du_i = \sum_{i=1}^n b_i du_i$. So, in particular, if we look at the (infinitesimal) coordinate parallelepiped around $\psi(u)$ (this is the image of the parallelepiped with sides $e_1 = (\Delta u_1, 0, \dots, 0)$, ..., $e_n = (0, \dots, 0, \Delta u_n)$ via ψ), its sides are $f_1 = b_1\Delta u_1, \dots, f_n = b_n \Delta u_n$. The Gram matrix of these vectors has components $\langle f_i, f_j\rangle = \langle b_i, b_j\rangle \Delta u_i \Delta u_j = g_{ij}\Delta u_i \Delta u_j$. It is easy to see that the square root of the determinant of this matrix is $\sqrt{\det(G)}\Delta u_1 \dots \Delta u_n$.

(c) To this end, we will have to make use of the change of variables formula for integrals in many dimensions: Recall from Analysis II that if $\Phi : x = (x_1, \dots, x_n) \to y = (y_1, \dots, y_n) = (\phi_1(x), \dots, \phi_n(x))$ is a change of variables, then for any domain $\mathcal{U} \subset \mathbb{R}^n$ and any continuous function f:

$$\int_{\Phi(\mathcal{U})} f(\phi^{-1}(y))\, dy_1 \dots dy_n = \int_{\mathcal{U}} f(x)|\det(D\Phi)| dx_1 \dots dx_n,$$

where $\det(D\Phi)$ is the determinant of the Jacobian of the transformation:

$$D\Phi = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & & \vdots \\ \vdots & & \ddots & \end{pmatrix}.$$

Note that since

$$D(\Phi^{-1})|_y = (D\Phi|_{\Phi^{-1}(y)})^{-1}$$

(this follows easily by differentiating the relation $\Phi \circ \Phi^{-1}(y) = y$), the above relation is equivalent to the following (as can be seen by applying the above relation to the function $f(x)/|\det(D\Phi)(x)|$ in place of $f(x)$):

$$\int_{\mathcal{U}} f(x)|\det(D\Phi)| dx_1 \dots dx_n = \int_{\Phi(\mathcal{U})} f(\Phi^{-1}(y))|\det(D(\Phi^{-1}))(y)|\, dy_1 \dots dy_n.$$

Recall also that, if $h : \Omega' \to \Omega$ is a C^1 diffeomorphism and G is the metric tensor associated to the parametrization $\psi : \Omega \to \mathcal{M}$, while G' is the metric tensor associated with the parametrization $\psi' = \psi \circ h : \Omega' \to \mathcal{M}$, then G and G' are related by

$$G'(x) = Dh^T(x) \cdot G(h(x)) \cdot Dh(x) \quad \text{for all } x \in \Omega'.$$

Therefore, we can immediately calculate after doing the change of variables $u' = h^{-1}(u)$ for $u \in \Omega$:

$$\int_\Omega \sqrt{\det(G(u))}\, du_1 \dots du_n \overset{u'=h^{-1}(u)}{=} \int_{h^{-1}(\Omega)} \sqrt{\det(G(h(u')))} |\det(Dh)(u')|\, du'_1 \dots du'_n$$

$$
\begin{aligned}
&= \int_{\Omega'} \sqrt{\det\left((Dh(u')^{-1})^T \cdot G'(u') \cdot Dh(u')^{-1}\right)}\,|\det(Dh)(u')|\, du'_1 \dots du'_n \\
&= \int_{\Omega'} \sqrt{(\det(Dh)(u')^{-1})^2 \cdot \det(G'(u'))}\,|\det(Dh)(u')|\, du'_1 \dots du'_n \\
&= \int_{\Omega'} \sqrt{\cdot \det(G'(u'))}\,|\det(Dh(u')^{-1})|\,|\det(Dh(u'))|\, du'_1 \dots du'_n \\
&= \int_{\Omega'} \sqrt{\cdot \det(G'(u'))}\, du'_1 \dots du'_n .
\end{aligned}
$$

10.3 Prove that the area of a regular parametrized surface $\psi : \Omega \to S \subset \mathbb{R}^3$ of class C^1 can be computed by the formula

$$\mathrm{Area}(S) = \iint_\Omega \left\| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right\| du\, dv.$$

Solution. Using Lagrange's identity

$$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b\rangle^2,$$

with the usual notation $b_1 = \frac{\partial \psi}{\partial u}, b_2 = \frac{\partial \psi}{\partial v}$, we obtain

$$\left\| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right\|^2 = g_{11} g_{22} - g_{12}^2.$$

Since, by definition,

$$\mathrm{Area}(S) = \int_\Omega \sqrt{\det(G)}\, du dv = \sqrt{g_{11} g_{22} - g_{12}^2}\, du\, dv,$$

we obtain the desired expression.

10.4 Let $f : [a, b] \to \mathbb{R}$ be a C^1 function and let S be the surface of revolution in $\mathbb{R}^3$ obtained by rotating the graph of f around the x-axis. Prove carefully that

$$\mathrm{Area}(S) = 2\pi \int_a^b \sqrt{1 + (f'(x))^2}\, |f(x)|\, dx.$$

Solution. We parametrize the surface by

$$\psi(x, \theta) = \big(x,\ f(x)\cos\theta,\ f(x)\sin\theta\big), \quad x \in [a, b], \theta \in [0, 2\pi)$$

Tangent vectors:

$$b_1 = (1, f'(x)\cos\theta, f'(x)\sin\theta), \qquad b_2 = (0, -f(x)\sin\theta, f(x)\cos\theta).$$

Metric (computed by $g_{ij} = \langle b_i, b_j \rangle$):

$$g_{11} = 1 + f'(x)^2, \qquad g_{12} = 0, \qquad g_{22} = f(x)^2.$$

Thus, we compute

$$\mathrm{Area}(S) = \int_a^b \int_0^{2\pi} \sqrt{\det(g)}\, d\theta dx = \int_a^b \int_0^{2\pi} \sqrt{g_{11}g_{22} - g_{12}^2}\, d\theta dx = \int_a^b \int_0^{2\pi} \sqrt{1 + f'(x)^2}\, |f(x)|\, d\theta dx = 2\pi \int_a^b \sqrt{}$$

10.5 **(a)** Give a maximal open domain on which polar coordinates define a diffeomorphism $\psi : (r, \theta) \mapsto (x, y)$.

(b) Compute the associated metric tensor.

(c) Deduce from it the formula for computing the area of a domain in polar coordinates.

Solution. (a) The map

$$\psi(r, \theta) = (r\cos\theta, r\sin\theta)$$

defines a diffeomorphism between

$$\Omega = \{(r, \theta) \mid r > 0,\ -\pi < \theta < \pi\}$$

and

$$U = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0 \text{ or } x > 0\} = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leqslant 0\}.$$

(b) We compute:

$$b_1 = \frac{\partial \psi}{\partial r} = (\cos\theta, \sin\theta), \qquad b_2 = \frac{\partial \psi}{\partial \theta} = (-r\sin\theta, r\cos\theta).$$

Thus

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = r^2.$$

So the metric is

$$G(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

(c) Let $\mathcal{V}$ be a domain in $\mathbb{R}^2$. Its area in polar coordinates should then be

$$\mathrm{Area}(\mathcal{V}) = \int_{\psi^{-1}(\mathcal{V})} \sqrt{\det(G(r, \theta))}\, dr d\theta = \int_{\psi^{-1}(\mathcal{V})} r\, dr d\theta.$$

As a sanity check, if $\mathcal{V}$ is the disc $\{x^2 + y^2 < R^2\}$ (which in polar coordinates is the set $\psi^{-1}(\mathcal{V}) = \{r < R\}$), we compute:

$$\mathrm{Area}(\mathcal{V}) = \int_{\psi^{-1}(\mathcal{V})} r\, dr d\theta = \int_0^{2\pi} \int_0^R r\, dr d\theta = \pi R^2$$

(as expected).

10.6 Consider the helicoid defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x \sin(z) - y \cos(z) = 0\}.$$

- **(a)** Prove that the helicoid is a ruled surface (see Ex. 9.5) and describe the geometry of this surface.
- **(b)** Show that the map
$$\psi(u, v) = (v \cos u,\ v \sin u,\ u)$$
defines a (global) diffeomorphism between $\mathbb{R}^2$ and S.
- **(c)** Compute the metric tensor associated with this parametrization

Solution. (a) The surface is defined by $S = f^{-1}(0)$ where $f(x, y, z) = x \sin z - y \cos z$. Its gradient never vanishes:

$$\nabla f = (\sin z, -\cos z, x\cos z + y \sin z), \|\nabla f\|^2 = \sin^2 z + \cos^2 z + (x \cos z + y \sin z)^2 = 1 + (x \cos z + y \sin z)^2 \geqslant 1,$$

so S is a regular 2-dimensional submanifold.

For fixed c, the intersection $\{z = c\} \cap S$ is given by

$$z = c, \quad x \sin c - y \cos c = 0,$$

which is a straight line in $\mathbb{R}^3$. Thus each point lies on a straight line contained in S, so S is ruled. Geometrically, a horizontal line rotates at constant speed while moving along the z-axis.

(b) The process we are going to follow in this exercise can be used in general to verify that a given map is a parametrization.

For the map $\psi(u, v)$ we compute

$$x(u, v) \sin(z(u, v)) - y(u, v) \cos(z(u, v)) = v \cos u \sin u - v \sin u \cos u = 0,$$

so $\psi(u, v) \in S$.

Moreover, $\psi : \mathbb{R}^2 \to S$ is $1-1$ and onto. To see this, the easiest way is to try to solve directly for the inverse map: Consider the map $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\phi(x, y, z) = \begin{cases} \left(z,\ \frac{x}{\cos z}\right), & \text{if } \cos(z) \neq 0, \\ \left(z,\ \frac{y}{\sin z}\right), & \text{if } \cos(z) = 0 \end{cases}$$

(note that this is a well defined map, since if $\cos(z) = 0$ then $\sin(z) \neq 0$). It is then easy to check that, for all $(x, y, z) \in S$ (i.e. satisfying $x \sin z = y \cos z$) we have

$$\psi \circ \phi(x, y, z) = (x, y, z) \quad \text{and} \quad \phi \circ \psi(u, v) = (u, v)$$

(so the map ψ is onto S) and for all $(u, v) \in \mathbb{R}^2$,

$$\phi \circ \psi(u, v) = (u, v)$$

(so the map ψ is 1-1). In particular, $\phi|_S = \psi^{-1}$.

Since we already established that ψ is $1-1$ and onto and we have constructed its inverse, in order to show that its a diffeomorphism (say of class C^1) we have to show that both ψ and ψ^1 are of class C^1. It is easy to check that ψ is in fact C^∞ (just from its explicit expression). Verifying the same for ψ^{-1} directly is much harder. Instead, we will use the inverse function theorem: It suffices to show that, at every point $p \in \mathbb{R}^2$, the differential $d\psi_p : T_p\mathbb{R}^2 \to T_{\psi(p)}S$ is surjective. If this is the case, then the inverse function theorem would imply that, locally around p, ψ is a diffeomorphism on its image; this implies, in particular, that the inverse ψ^{-1} is C^1 around $\psi(p)$. Since this is true at every p and $\psi(\mathbb{R}^2) = S$, we infer that ψ^{-1} is C^1 everywhere.

Since $\dim T_{\psi(p)}S = 2$, in order to show that $d\psi_p : T_p\mathbb{R}^2 \to T_{\psi(p)}S$ is surjective, it is enough to show that$d\psi_p$, when viewed as a linear map into $\mathbb{R}^3$ has rank 2, (i.e. that ψ is an immersion). We can easily compute that the Jacobian matrix of ψ is

$$D\psi(u,v) = \begin{pmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \\ 1 & 0 \end{pmatrix},$$

which has rank 2 everywhere.

Remark. From the above, it immediately follows that in fact ψ is a C^∞ diffeomorphism. This is a corollary of the following fact (which you can easily show by computing higher order derivatives of the inverse of a function): Let $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^n$ be open domains and $f : \mathcal{U} \to \mathcal{V}$ be a C^k function which is $1-1$ and onto. If f^{-1} is of class C^1, then it is also of class C^k.

(c) We compute:

$$b_1 = \frac{\partial\psi}{\partial u} = (-v\sin u,\, v\cos u,\, 1), \qquad b_2 = \frac{\partial\psi}{\partial v} = (\cos u,\, \sin u,\, 0).$$

Thus

$$g_{11} = 1 + v^2, \qquad g_{12} = 0, \qquad g_{22} = 1.$$

$$G(u,v) = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

10.7 The catenary is the graph of the hyperbolic cosine, i.e., the curve $\alpha(t) = (t, \cosh t)$. This also happens to be the shape of a chain hanging from its two endpoints (deriving the shape of such a chain is an easy exercise in the context of a physics course).

- **(a)** Show that the curvature of α is given by $\kappa(t) = 1/\cosh^2(t)$.
- **(b)** Compute the evolute of α (this curve is called a *tractrix*).
- **(c)** Compute the arc length of the catenary from the initial point $\alpha(0) = (0,1)$, then give the natural parametrization of α.

(d) The surface of revolution of the catenary around the x-axis is called a catenoid. Compute the metric tensor of the catenoid (preferably using the natural parametrization of the catenary).

Solution. (a) Using

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

for $x(t) = t, y(t) = \cosh t$, one obtains $\kappa(t) = 1/\cosh^2 t$.

(b) The evolute is

$$\beta(t) = \alpha(t) + \rho(t)N(t).$$

Here $\rho(t) = 1/\kappa(t) = \cosh^2 t$, and the normal vector is

$$N = \frac{(-\sinh t,\, 1)}{\cosh t}.$$

Thus

$$\beta(t) = (t - \cosh t \sinh t,\ 2\cosh t).$$

(c) The velocity vector is

$$\dot{\gamma}(t) = (1, \sinh(t))$$

So the speed is:

$$V_\alpha(t) = \cosh t.$$

Thus arclength:

$$s(t) = \int_0^t \cosh\tau \, d\tau = \sinh t.$$

So the natural parametrization is

$$s \mapsto \left(\sinh^{-1} s,\ \sqrt{1+s^2}\right) = \left(\log(s + \sqrt{1+s^2}),\ \sqrt{1+s^2}\right).$$

(d) The catenoid parametrization via revolution:

$$\psi(s,\theta) = \left(\log(s+\sqrt{1+s^2}),\ \sqrt{1+s^2}\cos\theta,\ \sqrt{1+s^2}\sin\theta\right).$$

Metric:

$$G(s,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1+s^2 \end{pmatrix}.$$

Remark. Note that the expression of the metric is the same as for the helicoid in the previous exercise (up to exchange of the coordinates)! In particular, the two surfaces are locally isometric (but not globally isometric).

10.8 Let $\mathcal{S}_a \subset \mathbb{R}^3$ be the sphere of radius $a > 0$ centered at the origin. The stereographic projection is the map

$$\pi : \mathcal{S}_a \setminus \{(0,0,a)\} \to \mathbb{R}^2$$

which sends a point $p = (x, y, z) \neq (0, 0, a)$ to the unique point q in the plane $\mathbb{R}^2$ such that the three points $(0, 0, a)$, p, and q are aligned (we view $\mathbb{R}^2$ as the plane $\{z = 0\}$ in $\mathbb{R}^3$).

Let $\psi : \mathbb{R}^2 \to \mathcal{S}_a$ denote the inverse stereographic projection.

- **(a)** Find an explicit formula for ψ and show that ψ is a regular parametrization of $\mathcal{S}_a \setminus \{(0, 0, a)\}$.
- **(b)** Compute the associated metric tensor.
- **(c)** Is this parametrization conformal?
- **(d)** Prove that stereographic projection defines a homeomorphism between the sphere and the Alexandrov compactification of $\mathbb{R}^2$.

  (The Alexandrov compactification of $\mathbb{R}^2$ is the set $\widehat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ with the topology for which every neighborhood of a point of $\mathbb{R}^2$ is also a neighborhood in $\widehat{\mathbb{R}}^2$, and the complements of compact subsets of $\mathbb{R}^2$ form a neighborhood basis of the point ∞.)

Remark. Sometimes one defines stereographic projection onto a plane other than the equatorial plane; in particular, one often projects onto the tangent plane at the “south pole” $(0, 0, -a)$.

Solution. (a) We organize the computation carefully. Let $\psi(u, v) = (x, y, z)$ be the point on the sphere S_a aligned with $(u, v, 0)$ and $(0, 0, a)$. Because stereographic projection is equivariant under rotation about the z–axis, it is convenient to denote

$$r = \sqrt{u^2 + v^2}, \qquad \rho = \sqrt{x^2 + y^2}.$$

From Thales’ theorem and the Pythagorean identity,

$$\frac{r}{a} = \frac{\rho}{a - z}, \qquad z^2 + \rho^2 = a^2.$$

Thus

$$\frac{r^2}{a^2} = \frac{\rho^2}{(a - z)^2} = \frac{a^2 - z^2}{(a - z)^2} = \frac{a + z}{a - z}.$$

Solving for z gives

$$z(u, v) = a\,\frac{r^2 - a^2}{r^2 + a^2} = a\,\frac{u^2 + v^2 - a^2}{u^2 + v^2 + a^2}.$$

Then

$$\rho = \frac{(a - z)r}{a} = \frac{2a^2\, r}{u^2 + v^2 + a^2},$$

and therefore

$$x(u, v) = \frac{\rho u}{r} = \frac{2a^2 u}{u^2 + v^2 + a^2}, \qquad y(u, v) = \frac{2a^2 v}{u^2 + v^2 + a^2}.$$

Let

$$\lambda(u,v) = \frac{1}{u^2+v^2+a^2}.$$

The parametrization becomes

$$\psi(u,v) = \lambda(u,v)\left(2a^2u,\ 2a^2v,\ a(u^2+v^2-a^2)\right).$$

One may verify directly that $\|\psi(u,v)\| = a$, so the image lies on S_a. Regularity will follow from the computation in part (b).

(b) We compute the metric tensor. First note:

$$\frac{\partial\lambda}{\partial u} = -2\lambda^2 u, \qquad \frac{\partial\lambda}{\partial v} = -2\lambda^2 v.$$

After calculation one finds the tangent vectors:

$$b_1 = \frac{\partial\psi}{\partial u} = 2a^2\lambda^2(a^2-u^2+v^2,\ -2uv,\ 2au),$$

$$b_2 = \frac{\partial\psi}{\partial v} = 2a^2\lambda^2(-2uv,\ a^2+u^2-v^2,\ 2av).$$

A direct computation shows

$$\langle b_1, b_2\rangle = 0.$$

Moreover,

$$g_{11} = \langle b_1, b_1\rangle = 4a^4\lambda^2, \qquad g_{22} = \langle b_2, b_2\rangle = 4a^4\lambda^2.$$

Thus the metric tensor is

$$G(u,v) = 4a^4\lambda(u,v)^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

or in differential form,

$$ds^2 = \frac{4a^4\left(du^2+dv^2\right)}{\left(a^2+u^2+v^2\right)^2}.$$

Because the tangent vectors b_1 and b_2 are linearly independent at all points, the parametrization is regular.

(c) The metric is a scalar multiple of the Euclidean metric, hence *angles are preserved.* Therefore the stereographic parametrization is conformal.

(d) From the definition of the Alexandrov compactification, it suffices to verify that

$$\lim_{(u,v)\to\infty}\psi(u,v) = (0,0,a).$$

Geometrically this is clear (points in the plane far away project close to the North pole), and it is also confirmed by the explicit formulas above.

Thus stereographic projection induces a homeomorphism

$$S_a \longrightarrow \widehat{\mathbb{R}^2}, \qquad (0,0,a) \leftrightarrow \infty.$$